Kruskal–Penrose Formalism
for Lightlike Thin-Shell Wormholes

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Abstract The original formulation of the "Einstein–Rosen bridge" in the classic paper of Einstein and Rosen (1935) is historically the first example of a *static* spherically-symmetric wormhole solution. It is *not equivalent* to the concept of the *dynamical* and *non-traversable* Schwarzschild wormhole, also called "Einstein–Rosen bridge" in modern textbooks on general relativity. In previous papers of ours we have provided a mathematically correct treatment of the original "Einstein–Rosen bridge" as a *traversable* wormhole by showing that it requires the presence of a special kind of "exotic matter" located on the wormhole throat – a lightlike brane (the latter was overlooked in the original 1935 paper). In the present note we continue our thorough study of the original "Einstein–Rosen bridge" as a simplest example of a lightlike thin-shell wormhole by explicitly deriving its description in terms of the Kruskal–Penrose formalism for maximal analytic extension of the underlying wormhole spacetime manifold. Further, we generalize the Kruskal–Penrose description to the case of more complicated lightlike thin-shell wormholes with two throats exhibiting a remarkable property of QCD-like charge confinement.

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1 Introduction

The principal object of study in the present note is the class of static spherically symmetric *lightlike thin-shell* wormhole solutions in general relativity, i.e., spacetimes with wormhole geometries and "throats" being lightlike ("null") hypersurfaces (for the importance and impact of lightlike hypersurfaces, see Refs. [1–3]). The explicit construction of lightlike thin-shell wormholes based on a self-consistent Lagrangian action formalism for the underlying lightlike branes occupying the wormhole "throats" and serving as material (and electrical charge) sources for the gravity to generate the wormhole spacetime geometry was given in a series of previous papers [4–8].[1]

The celebrated "Einstein–Rosen bridge", originally formulated in the classic paper [10], is historically the first and simplest example of a *static* spherically-symmetric wormhole solution – it is a 4-dimensional spacetime manifold consisting of two identical copies of the exterior Schwarzschild spacetime region matched (glued together) along their common horizon.

Let us immediately emphasize that the original construction in [10] of the "Einstein–Rosen bridge" is *not equivalent* to the notion of the *dynamical* Schwarzschild wormhole, also called "Einstein–Rosen bridge" in several standard textbooks (e.g. Ref. [11]), which employs the formalism of Kruskal–Szekeres maximal analytic extension of Schwarzschild black hole spacetime geometry. Namely, the two regions in Kruskal–Szekeres manifold corresponding to the outer Schwarzschild spacetime region beyond the horizon ($r > 2m$) and labeled (I) and (III) in Ref. [11] are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0$, $V = 0$ in Kruskal–Szekeres coordinates), whereas in the original Einstein–Rosen "bridge" construction the boundary between the two identical copies of the outer Schwarzschild space-time region ($r > 2m$) is a three-dimensional lightlike hypersurface ($r = 2m$). Physically, the most significant difference is that the "textbook" version of the "Einstein–Rosen bridge" (Schwarzschild wormhole) is *non-traversable*, i.e., there are *no* timelike or lightlike geodesics connecting points belonging to the two separate outer Schwarzschild regions (I) and (III). This is in sharp contrast w.r.t. the original Einstein–Rosen bridge (within its consistent formulation as a lightlike thin-shell wormhole [5]), which is a *traversable wormhole* (see also Sect. 3 below).

However, as explicitly demonstrated in Refs. [5, 6], the originally proposed in [10] Einstein–Rosen "bridge" wormhole solution *does not* satisfy the vacuum Einstein equations at the wormhole "throat". The mathematically consistent formulation of the original Einstein–Rosen "bridge" requires solving Einstein equations of bulk $D = 4$ gravity coupled to a lightlike brane with a well-defined world-volume action [12–15]. The lightlike brane locates itself automatically on the wormhole throat gluing together the two "universes" - two identical copies of the external spacetime region of a Schwarzschild black hole matched at their common horizon, with a special relation

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[1]For the general construction of *timelike* thin-shell wormholes, see the book [9].

between the (negative) brane tension and the Schwarzschild mass parameter. This is briefly reviewed in Sect. 2.

Traversability of the correctly formulated Einstein–Rosen bridge as a lightlike thin-shell wormhole is explicitly demonstrated in Sect. 3 in the sense of passing through the wormhole throat from the "left" to the "right" universe within *finite proper time* of a travelling observer.

In Sect. 4 we explicitly construct the Kruskal–Penrose maximal analytic extension of the proper Einstein–Rosen bridge wormhole manifold. In particular, the pertinent Kruskal–Penrose manifold involves a special identification of the future horizon of the "right" universe with the past horizon of the "left" universe, which is the mathematical manifestation of the wormhole traversability.

In Sect. 5 we extend our construction of Kruskal–Penrose maximal analytic extension of the total wormhole manifold to the case of a physically interesting wormhole solution with two "throats" which exhibits a remarkable property of charge and electric flux confinement [16] resembling the quark confinement property of quantum chromodynamics.

Section 6 contains our concluding remarks.

2 Einstein–Rosen Bridge as Lightlike Thin-Shell Wormhole

The Schwarzschild spacetime metric is the simplest static spherically symmetric black hole metric, written in standard coordinates (t, r, θ, φ) (e.g. [11]):

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad, \quad A(r) = 1 - \frac{r_0}{r}, \quad (1)$$

where $r_0 \equiv 2m$ (m – black hole mass parameter):

- $r > r_0$ defines the exterior spacetime region; $r < r_0$ is the black hole region;
- r_0 is the horizon radius, where $A(r_0) = 0$ ($r = r_0$ is a non-physical coordinate singularity of the metric (1), unlike the physical spacetime singularity at $r = 0$).

In constructing the maximal analytic extension of the Schwarzschild spacetime geometry – the *Kruskal–Szekeres* coordinate chart – essential intermediate use is made of the so called "tortoise" coordinate r^* (for light rays $t \pm r^* = \text{const}$):

$$\frac{dr^*}{dr} = \frac{1}{A(r)} \quad \longrightarrow \quad r^* = r + r_0 \ln|r - r_0| \,. \tag{2}$$

The Kruskal–Szekeres ("light-cone") coordinates (v, w) are defined as follows (e.g. [11]):

$$v = \pm\frac{1}{\sqrt{2k_h}}e^{k_h\left(t+r^*\right)} \quad, \quad w = \mp\frac{1}{\sqrt{2k_h}}e^{-k_h\left(t-r^*\right)} \tag{3}$$

with all combinations of the overall signs, where $k_h = \frac{1}{2}\partial_r A(r)\big|_{r=r_0} = \frac{1}{2r_0}$ is the so called "surface gravity" (related to the Hawking temperature as $\frac{k_h}{2\pi} = k_B T_{\text{hawking}}$). Equation (3) are equivalent to:

$$\mp vw = \frac{1}{k_h}e^{2k_h r^*} \quad , \quad \mp\frac{v}{w} = e^{2k_h t} \; , \tag{4}$$

wherefrom t and r^* are determined as functions of vw.

Depending on the combination of the overall signs Eq. (3) define a doubling the regions of the standard Schwarzschild geometry [11]:

  (i) $(+,-)$ – exterior Schwarzschild region $r > r_0$ (region I);
 (ii) $(+,+)$ – black hole $r < r_0$ (region II);
(iii) $(-,+)$ – second copy of exterior Schwarzschild region $r > r_0$ (region III);
 (iv) $(-,-)$ – "white" hole region $r < r_0$ (region IV).

The metric (1) becomes:

$$ds^2 = \widetilde{A}(vw)dvdw + r^2(vw)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad , \quad \widetilde{A}(vw) \equiv \frac{A\big(r(vw)\big)}{k_h^2 vw} \; , \tag{5}$$

so that now there is no coordinate singularity on the horizon ($v = 0$ or $w = 0$) upon using Eq. (2): $\widetilde{A}(0) = -4$.

In the classic paper [10] Einstein and Rosen introduced in (1) a new radial-like coordinate u via $r = r_0 + u^2$ and let $u \in (-\infty, +\infty)$:

$$ds^2 = -\frac{u^2}{u^2 + r_0}dt^2 + 4(u^2 + r_0)du^2 + (u^2 + r_0)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \; . \tag{6}$$

Thus, (6) describes two identical copies of the exterior Schwarzschild spacetime region $(r > r_0)$ for $u > 0$ and $u < 0$, which are formally glued together at the horizon $u = 0$.

Unfortunately, there are serious problems with (6):

- The Einstein–Rosen metric (6) has coordinate singularity at $u = 0$: $\det\|g_{\mu\nu}\|_{u=0} = 0$.
- More seriously, the Einstein equations for (6) acquire an ill-defined non-vanishing "matter" stress-energy tensor term on the r.h.s., which was overlooked in the original 1935 paper!

Indeed, as explained in [5], from Levi-Civita identity $R_0^0 = -\frac{1}{\sqrt{-g_{00}}}\nabla_{(3)}^2\left(\sqrt{-g_{00}}\right)$ we deduce that (6) solves vacuum Einstein equation $R_0^0 = 0$ for all $u \neq 0$. However, since $\sqrt{-g_{00}} \sim |u|$ as $u \to 0$ and since $\frac{\partial^2}{\partial u^2}|u| = 2\delta(u)$, Levi-Civita identity tells us that:

$$R_0^0 \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2) \; , \tag{7}$$

and similarly for the scalar curvature $R \sim \frac{1}{|u|}\delta(u) \sim \delta(u^2)$.

In [5] we proposed a correct reformulation of the original Einstein–Rosen bridge as a mathematically consistent traversable lightlike thin-shell wormhole introducing a different radial-like coordinate $\eta \in (-\infty, +\infty)$, by substituting $r = r_0 + |\eta|$ in (1):

$$ds^2 = -\frac{|\eta|}{|\eta| + r_0}dt^2 + \frac{|\eta| + r_0}{|\eta|}d\eta^2 + (|\eta| + r_0)^2 \left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) . \tag{8}$$

Equation (8) is the correct spacetime metric for the original Einstein–Rosen bridge:

- Equation (8) describes two "universes" – two identical copies of the exterior Schwarzschild spacetime region for $\eta > 0$ and $\eta < 0$.
- Both "universes" are correctly glued together at their common horizon $\eta = 0$. Namely, the metric (8) solves Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}^{(brane)} , \tag{9}$$

  where on the r.h.s. $T_{\mu\nu}^{(brane)} = S_{\mu\nu}\delta(\eta)$ is the energy-momentum tensor of a special kind of *lightlike brane* located on the common horizon $\eta = 0$ – the wormhole "throat".
- The lightlike analogues of W. Israel's junction conditions on the wormhole "throat" are satisfied [5, 6].
- The resulting lightlike thin-shell wormhole is traversable (see Sect. 3 below).

The energy-momentum tensor of lightlike branes $T_{\mu\nu}^{(brane)}$ is self-consistently derived as $T_{\mu\nu}^{(brane)} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{LL}}{\delta g^{\mu\nu}}$ from the following manifestly reparametrization invariant world-volume Polyakov-type lightlike brane action (written for arbitrary $D = (p+1) + 1$ embedding spacetime dimension and $(p+1)$-dimensional brane world-volume):

$$S_{LL} = -\frac{1}{2}\int d^{p+1}\sigma\, T b_0^{\frac{p-1}{2}} \sqrt{-\gamma}\left[\gamma^{ab}\bar{g}_{ab} - b_0(p-1)\right] , \tag{10}$$

$$\bar{g}_{ab} \equiv g_{ab} - \frac{1}{T^2}\left(\partial_a u + q\mathcal{A}_a\right)\left(\partial_b u + q\mathcal{A}_b\right) , \quad \mathcal{A}_a \equiv \partial_a X^\mu A_\mu . \tag{11}$$

Here and below the following notations are used:

- γ_{ab} is the *intrinsic* Riemannian metric on the world-volume with $\gamma = \det \|\gamma_{ab}\|$; b_0 is a positive constant measuring the world-volume "cosmological constant"; $(\sigma) \equiv (\sigma^a)$ with $a = 0, 1, \ldots, p$; $\partial_a \equiv \frac{\partial}{\partial\sigma^a}$.
- $X^\mu(\sigma)$ are the p-brane embedding coordinates in the bulk D-dimensional spacetime with Riemannian metric $g_{\mu\nu}(x)$ ($\mu, \nu = 0, 1, \ldots, D-1$). A_μ is a spacetime electromagnetic field (absent in the present case).

- $g_{ab} \equiv \partial_a X^\mu g_{\mu\nu}(X) \partial_b X^\nu$ is the *induced* metric on the world-volume which becomes *singular* on-shell – manifestation of the lightlike nature of the brane.
- u is auxiliary world-volume scalar field defining the lightlike direction of the induced metric and it is a non-propagating degree of freedom.
- T is *dynamical (variable)* brane tension (also a non-propagating degree of freedom).
- Coupling parameter q is the surface charge density of the LL-brane ($q = 0$ in the present case).

The Einstein Eq. (9) imply the following relation between the lightlike brane parameters and the Einstein–Rosen bridge "mass" ($r_0 = 2m$):

$$- T = \frac{1}{8\pi m} \;\; , \;\; b_0 = \frac{1}{4} \; , \tag{12}$$

i.e., the lightlike brane dynamical tension T becomes *negative* on-shell – manifestation of "exotic matter" nature.

3 Einstein–Rosen Bridge as Traversable Wormhole

As already noted in [5, 6] traversability of the original Einstein–Rosen bridge is a particular manifestation of the traversability of lightlike "thin-shell" wormholes.[2] Here for completeness we will present the explicit details of the traversability within the proper Einstein–Rosen bridge wormhole coordinate chart (8) which are needed for the construction of the pertinent Kruskal–Penrose diagram in Sect. 4.

The motion of test-particle ("observer") of mass m_0 in a gravitational background is given by the reparametrization-invariant world-line action:

$$S_{\text{particle}} = \frac{1}{2} \int d\lambda \left[\frac{1}{e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - e m_0^2 \right] , \tag{13}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$, e is the world-line "einbein" and in the present case $(x^\mu) = (t, \eta, \theta, \varphi)$.

For a static spherically symmetric background such as (8) there are conserved Noether "charges" – energy $\mathcal{E}$ and angular momentum $\mathcal{J}$. In what follows we will consider purely "radial" motion ($\mathcal{J} = 0$) so, upon taking into account the "mass-shell" constraint (the equation of motion w.r.t. e) and introducing the world-line proper-time parameter τ ($\frac{d\tau}{d\lambda} = e m_0$), the timelike geodesic equations (world-lines of massive point particles) read:

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[2]Subsequently, traversability of the Einstein–Rosen bridge has been studied using Kruskal–Szekeres coordinates for the Schwarzschild black hole [17], or the 1935 Einstein–Rosen coordinate chart (6) [18].

$$\left(\frac{d\eta}{d\tau}\right)^2 = \frac{\mathcal{E}^2}{m_0^2} - A(\eta) \ , \quad \frac{dt}{d\tau} = \frac{\mathcal{E}}{m_0 A(\eta)} \ , \quad A(\eta) \equiv \frac{|\eta|}{|\eta| + r_0} \ . \qquad (14)$$

where $A(\eta)$ is the "$-g_{00}$" component of the proper Einstein–Rosen bridge metric (8).

For a test-particle starting for $\tau = 0$ at initial position in "our" (right) universe $\eta_0 = \eta(0)$, $t_0 = t(0)$ and *infalling* towards the "throat" the solutions of Eq. (14) read:

$$\frac{\mathcal{E}}{2k_h m_0} \int_{2k_h\eta(\tau)}^{2k_h\eta_0} dy \sqrt{(1 + |y|)\left[\left(1 + \left(1 - \frac{m_0^2}{\mathcal{E}^2}\right)|y|\right]^{-1}} = \tau \ , \qquad (15)$$

$$\frac{1}{2k_h} \int_{2k_h\eta(\tau)}^{2k_h\eta_0} dy \frac{1}{|y|} \sqrt{(1 + |y|)\left[\left(1 + \left(1 - \frac{m_0^2}{\mathcal{E}^2}\right)|y|\right]} = t(\tau) - t_0 \ . \qquad (16)$$

- Equation (15) shows that the particle will cross the wormhole "throat" ($\eta = 0$) for a finite proper-time $\tau_0 > 0$:

$$\tau_0 = \frac{\mathcal{E}}{2k_h m_0} \int_0^{2k_h\eta_0} dy \sqrt{(1 + |y|)\left[\left(1 + \left(1 - \frac{m_0^2}{\mathcal{E}^2}\right)|y|\right]^{-1}} \ . \qquad (17)$$

- It will continue into the second (left) universe and reach any point $\eta_1 = \eta(\tau_1) < 0$ within another **finite** proper-time $\tau_1 > \tau_0$.
- On the other hand, from (16) it follows that $t(\tau_0 - 0) = +\infty$, i.e., from the point of view of a static observer in "our" (right) universe it will take infinite "laboratory" time for the particle to reach the "throat" – the latter appears to the static observer as a future black hole horizon.
- Equation (16) also implies $t(\tau_0 + 0) = -\infty$, which means that from the point of view of a static observer in the second (left) universe, upon crossing the "throat", the particle starts its motion in the second (left) universe from infinite past, so that it will take an infinite amount of "laboratory" time to reach the point $\eta_1 < 0$ – i.e. the "throat" now appears as a past black hole horizon.

In analogy with the usual "tortoise" coordinate r^* for the Schwarzschild black hole geometry (2) let us now introduce Einstein–Rosen bridge "tortoise" coordinate η^* (recall $r_0 = \frac{1}{2k_h}$):

$$\frac{d\eta^*}{d\eta} = \frac{|\eta| + r_0}{|\eta|} \quad \longrightarrow \quad \eta^* = \eta + \text{sign}(\eta)r_0 \ln|\eta| \ . \qquad (18)$$

Let us note here an important difference in the behavior of the "tortoise" coordinates r^* (2) and η^* (18) in the vicinity of the horizon. Namely:

$$r^* \to -\infty \quad \text{for} \ r \to r_0 \pm 0 \ , \qquad (19)$$

i.e., when r approaches the horizon either from above or from below, whereas when η approaches the horizon from above or from below:

$$\eta^* \to \mp\infty \quad \text{for} \ \ \eta \to \pm 0 \ . \tag{20}$$

For infalling/outgoing massless particles (light rays) Eqs. (15)–(18) imply:

$$t \pm \eta^* = \text{const} \ . \tag{21}$$

For infalling massive particles towards the "throat" ($\eta = 0$) starting at $\eta_0^+ > 0$ in "our" (right) universe and crossing into the second (left) universe, or starting in the second (left) universe at some $\eta_0^- < 0$ and crossing into the "our" (right) universe, we have correspondingly (replacing τ-dependence with functional dependence w.r.t. η using first Eq. (14)):

$$\big[t \pm \eta^*\big](\eta) = \frac{\pm 1}{2k_h} \int_{2k_h\eta}^{2k_h\eta_0^{\pm}} dy \left(1 + \frac{1}{|y|}\right) \left[\sqrt{(1+|y|)\Big[(1 + \big(1 - \frac{m_0^2}{\mathcal{E}^2}\big)|y|\Big]^{-1}} - 1\right] \tag{22}$$

4 Kruskal–Penrose Diagram for Einstein–Rosen Bridge

We now define the maximal analytic extension of original Einstein–Rosen wormhole geometry (8) via introducing Kruskal-like coordinates (v, w) as follows:

$$v = \pm\frac{1}{\sqrt{2k_h}}e^{\pm k_h(t+\eta^*)} \ , \quad w = \mp\frac{1}{\sqrt{2k_h}}e^{\mp k_h(t-\eta^*)} \ , \tag{23}$$

implying:

$$-vw = \frac{1}{2k_h}e^{\pm 2k_h\eta^*} \ , \quad -\frac{v}{w} = e^{\pm 2k_h t} \ . \tag{24}$$

Here and below η^* is given by (18).

- The upper signs in (23) and (24) correspond to region I ($v > 0$, $w < 0$) describing "our" (right) universe $\eta > 0$.
- The lower signs in (23) and (24) correspond to region II ($v < 0$, $w > 0$) describing the second (left) universe $\eta < 0$.

The metric (8) of Einstein–Rosen bridge in the Kruskal-like coordinates (23) reads:

$$ds^2 = \widetilde{A}(vw)dvdw + \widetilde{r}^2(vw)\big(d\theta^2 + \sin^2\theta d\varphi^2\big) , \tag{25}$$

$$\widetilde{r}(vw) = r_0 + |\eta(vw)| \quad \left(r_0 \equiv \frac{1}{2k_h}\right) ,$$

$$\widetilde{A}(vw) = \frac{A\big(\eta(vw)\big)}{k_h^2 vw} = -\frac{4e^{-2k_h|\eta(vw)|}}{1 + 2k_h|\eta(vw)|} , \tag{26}$$

where $\eta(vw)$ is determined from (24) and (18) as:

$$-vw = \frac{|\eta|}{2k_h}e^{2k_h|\eta|} \quad \longrightarrow \quad |\eta(vw)| = \frac{1}{2k_h}\mathcal{W}(-4k_h^2 vw) , \tag{27}$$

$\mathcal{W}(z)$ being the Lambert (product-logarithm) function $(z = \mathcal{W}(z)e^{\mathcal{W}(z)})$.

Using the explicit expression (18) for η^* in (24) we find for the metric (25) and (26):

- "Throats" (horizons) – at $v = 0$ or $w = 0$;
- In region I the "throat" $(v > 0, w = 0)$ is a future horizon $(\eta = 0, t \to +\infty)$, whereas the "throat" $(v = 0, w < 0)$ is a past horizon $(\eta = 0, t \to -\infty)$.
- In region II the "throat" $(v = 0, w > 0)$ is a future horizon $(\eta = 0, t \to +\infty)$, whereas the "throat" $(v < 0, w = 0)$ is a past horizon $(\eta = 0, t \to -\infty)$.

It is customary to replace Kruskal-like coordinates (v, w) (23) with compactified Penrose-like coordinates $(\bar{v}, \bar{w})$:

$$\bar{v} = \arctan(\sqrt{2k_h}\, v) \quad , \quad \bar{w} = \arctan(\sqrt{2k_h}\, w) , \tag{28}$$

mapping the various "throats" (horizons) and infinities to finite lines/points:

- In region I: future horizon $(0 < \bar{v} < \frac{\pi}{2}, \bar{w} = 0)$; past horizon $(\bar{v} = 0, -\frac{\pi}{2} < \bar{w} < 0)$.
- In region II: future horizon $(\bar{v} = 0, 0 < \bar{w} < \frac{\pi}{2})$; past horizon $(-\frac{\pi}{2} < \bar{v} < 0, \bar{w} = 0)$.
- i_0 – spacelike infinity $(t = \text{fixed}, \eta \to \pm\infty)$:
  $i_0 = (\frac{\pi}{2}, -\frac{\pi}{2})$ in region I; $i_0 = (-\frac{\pi}{2}, \frac{\pi}{2})$ in region II.
- $i_\pm$ – future/past timelike infinity $(t \to \pm\infty, \eta = \text{fixed})$:
  $i_+ = (\frac{\pi}{2}, 0)$, $i_- = (0, -\frac{\pi}{2})$ in region I; $i_+ = (0, \frac{\pi}{2})$, $i_- = (-\frac{\pi}{2}, 0)$ in region II.
- J_+ – future lightlike infinity $(t \to +\infty, \eta \to \pm\infty, t \mp \eta^* = \text{fixed})$:
  $J_+ = (\bar{v} = \frac{\pi}{2}, -\frac{\pi}{2} < \bar{w} < 0)$ in region I;
  $J_+ = (-\frac{\pi}{2} < \bar{v} < 0, \bar{w} = \frac{\pi}{2})$ in region II.
- J_- – past lightlike infinity $(t \to -\infty, \eta \to \pm\infty)$, $t \pm \eta^* = \text{fixed}$:
  $J_- = (0 < \bar{v} < \frac{\pi}{2}, \bar{w} = -\frac{\pi}{2})$ in region I:
  $J_- = (\bar{v} = -\frac{\pi}{2}, 0 < \bar{w} < \frac{\pi}{2})$ in region II.

For infalling light rays starting in region I and crossing into region II we have the lightlike geodesic $t + \eta^* = c_1 \equiv \text{const}$. Thus, according to (23) we must identify the
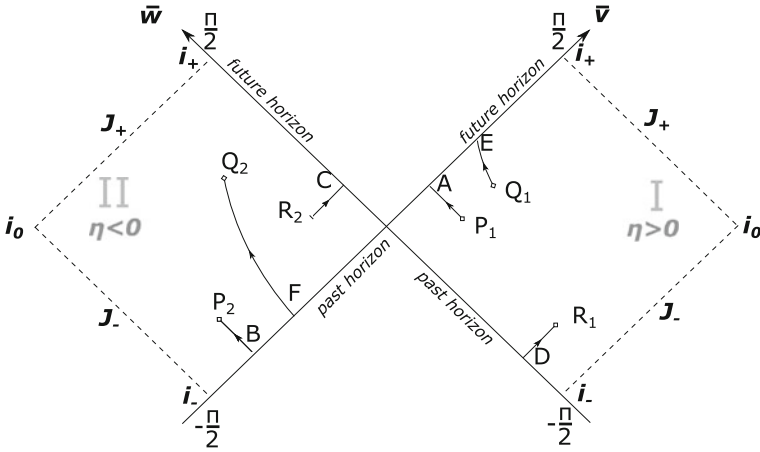
Fig. 1 Kruskal–Penrose diagram of the original Einstein–Rosen bridge

crossing point A on the future horizon of region I having Kruskal-like coordinates $(v = \frac{1}{\sqrt{2k_h}}e^{k_h c_1}, 0)$ with the point B on the past horizon of region II where the light rays enters into region II whose Kruskal-like coordinates are $(v = -\frac{1}{\sqrt{2k_h}}e^{-k_h c_1}, 0)$.

Similarly, for infalling light rays starting in region II and crossing into region I we have $t - \eta^* = c_2 \equiv$ const. Therefore, the crossing point C on the future horizon of region II having Kruskal-like coordinates $(0, w = \frac{1}{\sqrt{2k_h}}e^{k_h c_2})$ must be identified with the exit point D $(0, w = -\frac{1}{\sqrt{2k_h}}e^{-k_h c_2})$ on the past horizon of region I.

Inserting Eqs. (18)–(22) into the definitions of Kruskal-like (23) and Penrose-like (28) coordinates and taking into account the above identifications of horizons, we obtain the following visual representation of the Kruskal–Penrose diagram of the proper Einstein–Rosen bridge geometry (8) as depicted in Fig. 1:

- Future horizon in region I is identified with past horizon in region II as:

$$(\bar{v}, 0) \sim \left(\bar{v} - \frac{\pi}{2}, 0\right) . \tag{29}$$

Infalling light rays cross from region I into region II via paths $P_1 \to A \sim B \to P_2$ – all the way within finite world-line time intervals (the symbol $\sim$ means identification according to (29)). Similarly, infalling massive particles cross from region I into region II via paths $Q_1 \to E \sim F \to Q_2$ within finite proper-time interval.
- Future horizon in II is identified with past horizon in I:

$$(0, \bar{w}) \sim \left(0, \bar{w} - \frac{\pi}{2}\right) . \tag{30}$$

Infalling light rays cross from region II into region I via paths $R_2 \to C \sim D \to R_1$ where $C \sim D$ is identified according to (30).

5 Kruskal–Penrose Formalism for Two-Throat Lightlike Thin-Shell Wormhole

Now we will briefly discuss the extension of the construction of Kruskal–Penrose diagram for the proper Einstein–Rosen bridge wormhole to the case of lightlike "thin-shell" wormholes with two throats. To this end we will consider the physically interesting example of the charge-confining two-throat "tube-like" wormhole studied in [16]. It is a solution of gravity interacting with a special non-linear gauge field system and both coupled to a pair of oppositely charged lightlike branes (cf. Eqs. (10) and (11) above).

The full wormhole spacetime consists of three "universes" glued pairwise via the two oppositely charged lightlike branes located on their common horizons:

- Region I: right-most non-compact electrically neutral "universe" – exterior region beyond the Schwarzschild horizon of a Schwarzschild-de Sitter black hole;
- Region II: middle "tube-like" "universe" of Levi-Civita–Bertotti–Robinson type [19–21] with finite radial-like spacial extend and compactified transverse spacial dimensions;
- Region III: left-most non-compact electrically neutral "universe" – exterior region beyond the Schwarzschild horizon of a Schwarzschild-de Sitter black hole, mirror copy of the left-most "universe".
- Most remarkable property is that the whole electric flux generated by the two oppositely charged lightlike branes sitting on the two "throats" is *completely confined* within the finite-spacial-size middle "tube-like" universe – analog of QCD quark confinement!

For a visual representation, see Fig. 2 [16].

Generically, the metric of a spherically symmetric traversable lightlike thin-shell wormhole with two "throats" reads [16] ($-\infty < \eta < \infty$):

$$ds^2 = -A(\eta)dt^2 + \frac{d\eta^2}{A(\eta)} + r^2(\eta)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \;,$$

(31)

$$A(\eta_1) = 0 \;, \;\; A(\eta_2) = 0 \;, \;\; a_{(\pm)}^{(1)} = \pm\frac{\partial}{\partial\eta}A \big|_{\eta_1\pm0} > 0 \;, \;\; a_{(\pm)}^{(2)} = \pm\frac{\partial}{\partial\eta}A \big|_{\eta_2\pm0} > 0 \;.$$

Accordingly, for the wormhole "tortoise" coordinate η^* defined as in first Eq. (18) we have in the vicinity of the two horizons $\eta_{1,2}$:
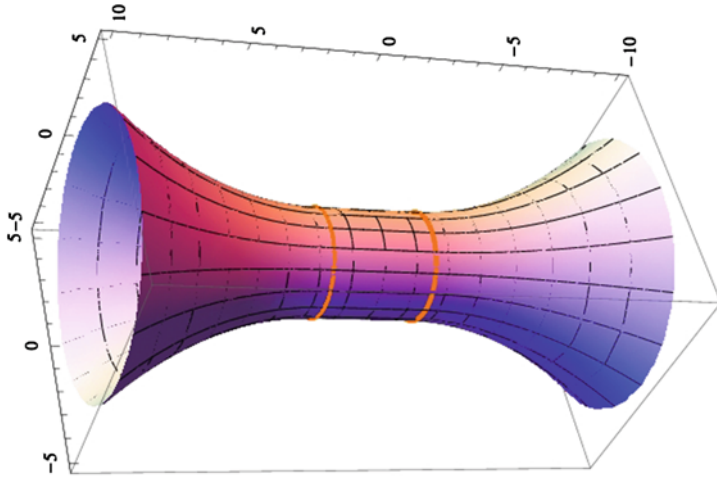
Fig. 2 Shape of $t = const$ and $\theta = \frac{\pi}{2}$ slice of charge-confining wormhole geometry. The whole electric flux is confined within the middle *cylindric* "tube" (region II) connecting the two infinite "funnels" (region I and region III). The *rings* on the edges of the "tube" depict the two oppositely charged lightlike branes

$$\eta^* = \text{sign}(\eta - \eta_1)a^{(1)}_{(\pm)} \ln|\eta - \eta_1| + O\big((\eta - \eta_1)^2\big) , \tag{32}$$

$$\eta^* = \text{sign}(\eta - \eta_2)a^{(2)}_{(\pm)} \ln|\eta - \eta_2| + O\big((\eta - \eta_2)^2\big) . \tag{33}$$

Now we can introduce the Kruskal-like and the compactified Kruskal–Penrose coordinates $(\bar{v}, \bar{w})$ for the maximal analytic extension of the two-throat lightlike thin-shell wormhole generalizing formulas (23) and (28) as follows:

- In region I (right-most universe) – $(+\infty > \eta > \eta_1)$:

$$\bar{v}, \bar{w} = \pm \frac{\pi}{2\sqrt{a^{(1)}_{(-)}}} \pm \frac{1}{\sqrt{a^{(1)}_{(+)}}} \arctan\left(e^{\frac{1}{2}a^{(1)}_{(+)}(\eta^* \pm t)}\right) \tag{34}$$

- In region II (middle universe) – $(\eta_1 > \eta > \eta_2)$; here $a^{(1)}_{(-)} = a^{(2)}_{(+)}$ which is satisfied in the case of the charge-confining two-throat "tube" wormhole:

$$\bar{v}, \bar{w} = \pm \frac{1}{\sqrt{a^{(1)}_{(-)}}} \arctan\left(e^{\frac{1}{2}a^{(1)}_{(-)}(\eta^* \pm t)}\right) . \tag{35}$$

- In region III (left-most universe) – $(\eta_2 > \eta > -\infty)$:

$$\bar{v}, \bar{w} = \mp \frac{\pi}{2\sqrt{a^{(2)}_{(-)}}} \pm \frac{1}{\sqrt{a^{(2)}_{(-)}}} \arctan\left(e^{\frac{1}{2}a^{(2)}_{(-)}(\eta^* \pm t)}\right) . \tag{36}$$
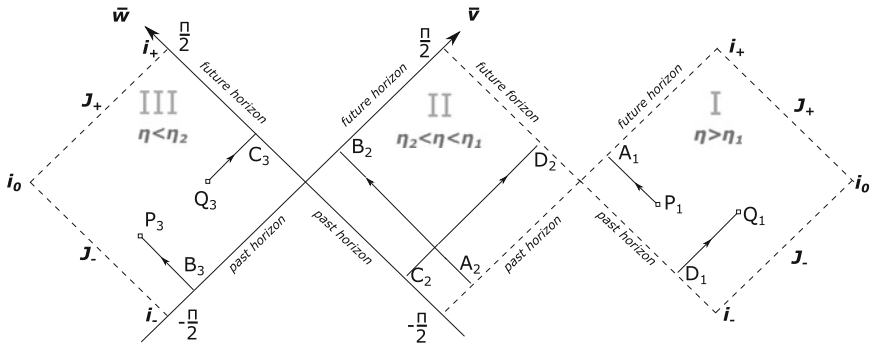
Fig. 3 Kruskal–Penrose diagram of "charge-confining" two-throat wormhole

The resulting Kruskal–Penrose diagram is depicted on Fig. 3.

In particular, infalling light ray starting in region I arrives in region III within finite world-line time interval ("proper-time" in the case of massive particle) on the path $P_1 \rightarrow A_1 \sim A_2 \rightarrow B_2 \sim B_3 \rightarrow P_3$, where the symbol $\sim$ indicates identification of the pertinent future and past horizons of the "glued" together neighboring "universes" analogous to the identification (29), (30) in the simpler case of Einstein–Rosen one-throat wormhole.

And similarly for an infalling light ray starting in region III and arriving in region I within finite world-line time interval on the path $Q_3 \rightarrow C_3 \sim C_2 \rightarrow D_2 \sim D_1 \rightarrow Q_1$.

6   Conclusions

The mathematically correct reformulation [5] of original Einstein–Rosen "bridge" construction, briefly reviewed in Sect. 2 above, shows that it is the simplest example in the class of static spherically symmetric *traversable* lightlike "thin-shell" wormhole solutions in general relativity. The consistency of Einstein–Rosen "bridge" as a traversable wormhole solution is guaranteed by the remarkable special properties of the world-volume dynamics of the lightlike brane, which serves as an "exotic" thin-shell matter (and charge) source of gravity.

In the present note we have explicitly derived the Kruskal-like extension and the associated Kruskal–Penrose diagram representation of the mathematically correctly defined original Einstein–Rosen "bridge" [5] with the following significant differences w.r.t. Kruskal–Penrose extension of the standard Schwarzschild black hole defining the corresponding "textbook" version of Einstein–Rosen "bridge" (the Schwarzschild wormhole) [11]:

- The pertinent Kruskal–Penrose diagram for the proper Einstein–Rosen bridge (Fig. 1) has only two regions corresponding to "our" (right) and the second (left) "universes" unlike the four regions in the standard Schwarzschild black hole case (no black/white hole regions).
- The proper original Einstein–Rosen bridge is a *traversable* static spherically symmetric wormhole unlike the non-traversable non-static "textbook" version. Traversability is equivalent to the pairwise specific identifications of future with past horizons of the neighboring Kruskal regions.

We have also extended the Kruskal–Penrose diagram construction to the case of lightlike "thin-shell" wormholes with two throats.

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